Boussinesq Solitary–Wave as a Multiple–Time Solution of the Korteweg–de Vries Hierarchy

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Abstract

We study the Boussinesq equation from the point of view of a multiple-time reductive perturbation method. As a consequence of the elimination of the secular producing terms through the use of the Korteweg–de Vries hierarchy, we show that the solitary–wave of the Boussinesq equation is a solitary–wave satisfying simultaneously all equations of the Korteweg–de Vries hierarchy, each one in an appropriate slow time variable.

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I. INTRODUCTION

As is well known, the Boussinesq model equation

$$u_{tt} - u_{xx} + u_{xxxx} - 3(u^2)_{xx} = 0 \,, \tag{1}$$

where $u(x,t)$ is a one–dimensional field and the subscripts denote partial differentiation, is completely integrable.[1] It is considered as an intermediate long–wave equation since its long–wave limit with a further restriction to waves moving in only one direction yields, at the lowest order, the Korteweg–de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0 \,, \tag{2}$$

which is an equation governing general weak nonlinear long–wave dynamics of dispersive systems.[2] Equation (1) has N–soliton solutions. In particular, its solitary–wave solution is of the form[3]

$$u = -2k^2 \operatorname{sech}^2 \left[k \left(x - \sqrt{1 - 4k^2} \, t \right) \right] \,, \tag{3}$$

where k is the wavenumber. Accordingly, its long–wave limit is related, also at the lowest order, to the solitary–wave solution of the KdV equation.

In a recent work,[4] we have considered a perturbative scheme based on the reductive perturbation method of Taniuti,[5] modified by the introduction of an infinite number of slow time-variables, which were given by $\tau_3 = \epsilon^3 t$, $\tau_5 = \epsilon^5 t$, etc. Then, we have shown that, as a consequence of a natural compatibility condition, a wave field satisfying the KdV equation in the time τ_3 must also satisfy all equations of the KdV hierarchy,[6] each one in a different slow time variable. The main reason for introducing these time variables was that they allowed for the construction of a perturbative series, valid for weak nonlinear dispersive systems, which was free of solitary–wave related secularities.

As stated above, the solitary–wave solution of the Boussinesq equation tends to the KdV solitary–wave for small wave–numbers. What we will show here is that, by making use of the perturbative scheme with multiple slow time–scales, the solitary–wave of the Boussinesq

equation may be written, in the slow variables, as a solitary–wave solution to the whole set of equations of the KdV hierarchy, each one in a different time–scale. This result follows both, from the general long–wave perturbation theory, and from the observation that the perturbative series truncates for a solitary–wave solution to the KdV hierarchy equations, rendering thus an exact solution for the Boussinesq equation.

The paper is organized as follows. In Section II the multiple–time formalism is introduced for the Boussinesq equation, and the first few evolution equations are obtained. In Section III we discuss how the KdV hierarchy equations show up, and in Section IV we show how they can be used to eliminate the solitary–wave related secularities of the evolution equations for the higher–order terms of the wave field. In Section V, by returning from the slow to the laboratory coordinates, we obtain the above mentioned relation between the solitary–waves of the Boussinesq and the KdV hierarchy equations, namely that the solitary–wave of the Boussinesq equation may be written, in the slow variables, as a solitary–wave satisfying simultaneously all equations of the KdV hierarchy . And finally, in Section VI, we summarize and discuss the results obtained.

II. THE MULTIPLE TIME FORMALISM

In order to study the long-wave limit of eq.(1), we will introduce slow space and time variables based on the long–wave limit of the linear dispersion relation

$$\omega = k \left(1 + k^2\right)^{1/2} . \tag{4}$$

This limit corresponds to take

$$k = \epsilon\kappa , \tag{5}$$

with ϵ a small parameter. Expanding the dispersion relation (4), the solution of the corresponding linear Boussinesq equation is given simply by

$$u = a \exp i \left[\epsilon\kappa(x - t) - \frac{\epsilon^3\kappa^3}{2}t + \frac{\epsilon^5\kappa^5}{8}t - \frac{\epsilon^7\kappa^7}{16}t + \cdots\right] . \tag{6}$$

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Based on this solution, we define now the slow space coordinate

$$\xi = \epsilon(x - t), \tag{7}$$

and the infinite sequence of slow time coordinates

$$\tau_3 = -\frac{\epsilon^3 t}{2} \quad ; \quad \tau_5 = \frac{\epsilon^5 t}{8} \quad ; \quad \tau_7 = -\frac{\epsilon^7 t}{16} \quad ; \quad \cdots \quad . \tag{8}$$

Accordingly, we have that

$$\frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial \xi} \tag{9}$$

and

$$\frac{\partial}{\partial t} = -\epsilon \frac{\partial}{\partial \xi} - \frac{\epsilon^3}{2} \frac{\partial}{\partial \tau_3} + \frac{\epsilon^5}{8} \frac{\partial}{\partial \tau_5} - \frac{\epsilon^7}{16} \frac{\partial}{\partial \tau_7} + \cdots \quad . \tag{10}$$

Notice that in the definition of τ_{2n+1}, we have already assumed specific slow time normalizations, as inspired by the long–wave expansion of the dispersion relation. As we are going to see, these normalizations are exactly those necessary to cancel out the solitary–wave related secularities appearing in the higher order evolution equations.

Returning to the nonlinear problem, we make now the expansion

$$u = \epsilon^2 \hat{u} = \epsilon^2 \left(u_0 + \epsilon^2 u_2 + \epsilon^4 u_4 + \cdots \right), \tag{11}$$

and we suppose that $u_{2n} = u_{2n}(\xi, \tau_3, \tau_5, ...)$, $n = 0, 1, 2, \ldots$, which corresponds to an extention in the sense of Sandri.[7] Substituting it, together with eqs.(9) and (10), into the Boussinesq equation (1), the resulting expression, up to terms of order ϵ^4, is:

$$\left[\frac{\partial^2}{\partial \xi \partial \tau_3} + \frac{\partial^4}{\partial \xi^4} + \frac{\epsilon^2}{4} \left(\frac{\partial^2}{\partial \tau_3{}^2} - \frac{\partial^2}{\partial \xi \partial \tau_5} \right) + \frac{\epsilon^4}{8} \left(\frac{\partial^2}{\partial \xi \partial \tau_7} - \frac{\partial^2}{\partial \tau_3 \partial \tau_5} \right) + \cdots \right] \hat{u}$$
$$-3 \frac{\partial^2}{\partial \xi^2} \left[(u_0)^2 + 2\epsilon^2 u_0 u_2 + \epsilon^4 (2u_0 u_4 + (u_2)^2) + \cdots \right] = 0. \tag{12}$$

We now proceed to an order–by–order analysis of the problem. At order ϵ^0 we get

$$\frac{\partial}{\partial \xi} \left[\frac{\partial u_0}{\partial \tau_3} - 3 \frac{\partial}{\partial \xi} (u_0)^2 + \frac{\partial^3 u_0}{\partial \xi^3} \right] = 0. \tag{13}$$

Integrating once and assuming a vanishing integration constant, we obtain

$$\frac{\partial u_0}{\partial \tau_3} - 6u_0 \frac{\partial u_0}{\partial \xi} + \frac{\partial^3 u_0}{\partial \xi^3} = 0 \,, \tag{14}$$

which is the KdV equation.

At order ϵ^2, eq.(12) yields

$$\frac{\partial}{\partial \xi} \left[\frac{\partial u_2}{\partial \tau_3} - 6\frac{\partial}{\partial \xi}(u_0 u_2) + \frac{\partial^3 u_2}{\partial \xi^3} \right] = \frac{1}{4}\frac{\partial^2 u_0}{\partial \xi \partial \tau_5} - \frac{1}{4}\frac{\partial^2 u_0}{\partial \tau_3{}^2} \,. \tag{15}$$

Using eq.(14), integrating once in ξ and assuming a vanishing integration constant, we obtain

$$\frac{\partial u_2}{\partial \tau_3} - 6\frac{\partial}{\partial \xi}(u_0 u_2) + \frac{\partial^3 u_2}{\partial \xi^3} = \frac{1}{4}\frac{\partial u_0}{\partial \tau_5} - \frac{1}{4}\frac{\partial^5 u_0}{\partial \xi^5} + 3u_0 \frac{\partial^3 u_0}{\partial \xi^3} + \frac{9}{2}\frac{\partial u_0}{\partial \xi}\frac{\partial^2 u_0}{\partial \xi^2} - 9(u_0)^2 \frac{\partial u_0}{\partial \xi} \quad . \tag{16}$$

Equation (16), as it stands, presents two problems. First, as $u_{0\tau_5}$ is not known *a priori*, it cannot be solved for u_2. In the next section we will show how to obtain the evolution of u_0 in the time τ_5 independently. The second problem is that the term $(\partial^5 u_0/\partial \xi^5)$, as a source term for u_2, is a secular producing term when u_0 is chosen to be a solitary–wave solution of the KdV equation. For instance, if we take the solution of eq.(14) proportional to $[\text{sech}^2 \theta]$, then $(\partial^5 u_0/\partial \xi^5)$ will contain a term proportional to $[\text{sech}^2 \theta \tanh \theta]$. Being a solution of the homogeneous part of eq.(16), this term produces a resonance, giving rise to non–uniformities in the perturbative series. It will turn out, however, that $u_{0\tau_5}$ can be used to cancel out this secular term.

III. THE RISE OF THE KORTEWEG–DE VRIES HIERARCHY

As we have seen, the field u_0 satisfies the KdV equation in the time τ_3:

$$u_{0\tau_3} = -u_{0\xi\xi\xi} + 6u_0\, u_{0\xi} \equiv F_3 \,. \tag{17}$$

The evolution of the same field u_0 in any of the higher–order times τ_{2n+1} can then be obtained in the following way.[4] First, to have a well ordered perturbative scheme we impose that each one of the equations

$$u_{0\tau_{2n+1}} = F_{2n+1}(u_0, u_{0\xi}, \dots) \tag{18}$$

be ϵ–independent when passing from the slow (u_0, ξ, τ_{2n+1}) to the laboratory coordinates (u, x, t). This step selects all possible terms to appear in $F_{2n+1}(u_0, u_{0\xi}, \dots)$. For instance, the evolution of u_0 in time τ_5 is restricted to be of the form

$$u_{0\tau_5} = \alpha u_{0(5\xi)} + \beta u_0 u_{0\xi\xi\xi} + (\beta + \gamma) u_{0\xi} u_{0\xi\xi} + \delta u_0^2 u_{0\xi}, \tag{19}$$

where α, β, γ and δ are constants. Then, by imposing the natural (in the multiple time formalism) compatibility condition[4]

$$\left(u_{0\tau_3}\right)_{\tau_{2n+1}} = \left(u_{0\tau_{2n+1}}\right)_{\tau_3}, \tag{20}$$

or equivalently,

$$\left(F_3\right)_{\tau_{2n+1}} = \left(F_{2n+1}\right)_{\tau_3}, \tag{21}$$

with F_3 given by eq.(17), it is possible to determine any F_{2n+1}. As it can be verified,[4] the resulting equations are those given by the KdV hierarchy. In particular, for $u_{0\tau_5}$ and $u_{0\tau_7}$ we obtain respectively

$$u_{0\tau_5} = u_{0(5\xi)} - 10u_0 u_{0\xi\xi\xi} - 20u_{0\xi} u_{0\xi\xi} + 30(u_0)^2 u_{0\xi}, \tag{22}$$

and

$$u_{0\tau_7} = -u_{0(7\xi)} + 14u_0 u_{0(5\xi)} + 42u_{0\xi} u_{0(4\xi)} +$$
$$+ 70u_{0\xi\xi} u_{0\xi\xi\xi} - 280u_0 u_{0\xi} u_{0\xi\xi} - 70(u_{0\xi})^3 - 70(u_0)^2 u_{0\xi\xi\xi}. \tag{23}$$

In principle, one could have an arbitrary constant multiplying the right-hand side of eqs.(22) and (23), which would correspond to an arbitrariness in the slow time normalizations. However, as we will see in the next section, the definition of the slow time variables we took implies that these constants must be chosen to be one, since in this case the perturbation theory is automatically rendered free of secularities. This choice is also the one which makes the linear limit of the perturbation theory compatible with the linear theory coming directly from eq.(1).

IV. HIGHER ORDER EVOLUTION EQUATIONS

From this point on, we are going to consider some specific solutions to our equations. First of all, we assume the solution of the KdV equation (17) to be the solitary–wave solution

$$u_0 = -2\kappa^2 \text{sech}^2 \left[\kappa(\xi - 4\kappa^2 \tau_3) + \theta \right] , \tag{24}$$

where θ is a phase. Moreover, since u_0 must satisfy also the equations of the KdV hierarchy, we assume that u_0 given by (24) be also a solitary–wave solution to all equations of the KdV hierarchy, each one in a different slow–time variable. This means that u_0 is actually

$$u_0 = -2\kappa^2 \text{sech}^2 \left[\kappa\xi - 4\kappa^3 \tau_3 + 16\kappa^5 \tau_5 - 64\kappa^7 \tau_7 + \cdots \right] . \tag{25}$$

We return now to eq.(16) for u_2. Substituting $u_{0\tau_5}$ from eq.(22), we obtain

$$u_{2\tau_3} - 6(u_0 u_2)_\xi + u_{2\xi\xi\xi} = \frac{1}{2} \left[-3(u_0)^2 u_{0\xi} + u_0 u_{0\xi\xi\xi} - u_{0\xi} u_{0\xi\xi} \right] . \tag{26}$$

In passing we notice that the use of the KdV hierarchy equation to express $u_{0\tau_5}$ automatically canceled out the secular–producing term $u_{0(5\xi)}$. Moreover, using the solitary–wave solution (25) for u_0, we see that the right–hand side of eq.(26) vanishes, leading to

$$u_{2\tau_3} - 6(u_0 u_2)_\xi + u_{2\xi\xi\xi} = 0 , \tag{27}$$

which is the linearized KdV equation. We will assume for it the trivial solution

$$u_2 = 0 . \tag{28}$$

With this result, order ϵ^2 is solved for the particular case we chose.

At order ϵ^4, and already assuming that $u_2 = 0$, eq.(12) gives

$$u_{4\tau_3\xi} - 6(u_0 u_4)_{\xi\xi} + u_{4(4\xi)} = \frac{1}{8} \left[-u_{0\tau_7\xi} + u_{0\tau_3\tau_5} \right] . \tag{29}$$

Using equations (17) and (22) to express $u_{0\tau_3}$ and $u_{0\tau_5}$ respectively, and integrating once in ξ, we obtain

$$u_{4\tau_3} - 6(u_0 u_4)_\xi + u_{4\xi\xi\xi} = \frac{1}{8}\Big[-u_{0\tau\tau} - u_{0(7\xi)} + 16u_0 u_{0(5\xi)} - 90(u_0)^2 u_{0\xi\xi\xi} +$$
$$+ 70u_{0\xi\xi}u_{0\xi\xi\xi} + 40u_{0\xi}u_{0(4\xi)} - 300u_0 u_{0\xi}u_{0\xi\xi} + 180(u_0)^3 u_{0\xi} - 60(u_{0\xi})^3 \Big]. \tag{30}$$

The term $u_{0(7\xi)}$ is the only resonant, that is, secular producing term to the solution u_4. Then, in the very same way we did before, we use the KdV hierarchy equation (23) to express $u_{0\tau\tau}$. After we do that, the secular producing term is automatically canceled out, and eq.(30) becomes

$$u_{4\tau_3} - 6(u_0 u_4)_\xi + u_{4\xi\xi\xi} = -2\big[u_{0\xi}u_{0(4\xi)} - u_0 u_{0(5\xi)} +$$
$$+ 10u_0 u_{0\xi}u_{0\xi\xi} - 5(u_{0\xi})^3 + 10(u_0)^2 u_{0\xi\xi\xi} - 20(u_0)^3 u_{0\xi}\big]. \tag{31}$$

Substituting again the solitary–wave solution (25) for u_0, we can easily see that the nonhomogeneous term of eq.(31) vanishes, leading to

$$u_{4\tau_3} - 6(u_0 u_4)_\xi + u_{4\xi\xi\xi} = 0. \tag{32}$$

And again, we take the trivial solution

$$u_4 = 0. \tag{33}$$

It is easy to see that this is a general result that will repeat at any higher order: for $n \geq 1$, the evolution of u_{2n} in the time τ_3, after using the KdV hierarchy equation to express $u_{0\tau_{2n+1}}$ and then substituting the solitary–wave solution (25) for u_0, is given by a homogeneous linearized KdV equation. Consequently, the solution

$$u_{2n} = 0 \quad ; \quad n \geq 1 \tag{34}$$

can be assumed for any higher order.

V. BACK TO THE LABORATORY COORDINATES

Let us now take the solitary–wave solution to all equations of the KdV hierarchy,

$$u_0 = -2\kappa^2\text{sech}^2\left[\kappa\xi - 4\kappa^3\tau_3 + 16\kappa^5\tau_5 - 64\kappa^7\tau_7 + \cdots\right], \tag{35}$$

and rewrite it in the laboratory coordinates. First, recall that we have made the expansion

$$u = \epsilon^2\hat{u} = \epsilon^2\left(u_0 + \epsilon^2 u_2 + \epsilon^4 u_4 + \cdots\right). \tag{36}$$

Thereafter, we have found a particular solution in which

$$u_{2n} = 0 \quad ; \quad n \geq 1. \tag{37}$$

Consequently, expansion (36) truncates leading to an exact solution of the form

$$u = \epsilon^2 u_0, \tag{38}$$

with u_0 given by eq.(35). Moreover, from eq.(5) we see that the wave–number κ is written in terms of the corresponding laboratory one by

$$\kappa = \epsilon^{-1}k. \tag{39}$$

Finally, the slow coordinates ξ and τ_{2n+1} are related to the laboratory ones, x and t, according to eqs.(7) and (8). Then, in the laboratory coordinates, the exact solution (38) is written as

$$u = -2k^2\text{sech}^2 k\left[x - \left(1 - 2k^2 - 2k^4 - 4k^6 + \cdots\right)t\right]. \tag{40}$$

Now, the series appearing inside the parenthesis can be summed, with the result

$$1 - 2k^2 - 2k^4 - 4k^6 + \cdots = \left(1 - 4k^2\right)^{1/2}. \tag{41}$$

Therefore, we get

$$u = -2k^2\text{sech}^2\left[k\left(x - \sqrt{1 - 4k^2}\, t\right)\right], \tag{42}$$

which is the well known solitary–wave solution of the Boussinesq equation (1).

VI. FINAL COMMENTS

By applying a multiple time version of the reductive perturbation method of Taniuti[5] to the Boussinesq model equation, and by eliminating the solitary–wave related secular producing terms through the use of the KdV hierarchy equations,[4] we have suceeded in establishing a relation between the solitary–wave satisfying all the equations of the KdV hierarchy and that of the Boussinesq equation. In other words, we have shown that the solitary–wave of the Boussinesq equation is given, in slow variables, by the solitary–wave satisfying simultaneously all the equations of the KdV hierarchy. Accordingly, while the KdV solitary–wave depends only on one slow time variable, namely τ_3, the solitary–wave of the Boussinesq equation can be thought as depending on the infinite sequence of slow time variables.

The above considerations put in evidence the universal character played not only by KdV, but by all equations of the KdV hierarchy in relation to general weak nonlinear dispersive systems. For such systems, as we have already said, it is always possible to define slow variables in which the KdV equation emerges at the lowest relevant order of the reductive perturbation method, and consequently, compatibility and secularity–free requirements imply that all equations of the KdV hierarchy emerge as well. In the case where the perturbative series truncates, we may then obtain an exact solution of the original equation, which is, in this sense, reconstructed from the perturbative expansion. Of course, the Boussinesq solitary–wave is a well-known solution, but not the least, it remains that a possible method of construction of solutions for more involved system of equations can be envisaged. The return to the laboratory coordinates, then, make the connection between a solution of the KdV hierarchy and that of the equations governing the original system.

To conclude, we may conjecture that, whenever the original model equation has an exact solitary–wave solution, the series may somehow be truncated (or eventually summed) and a relation will be established between the solitary–wave of the KdV hierarchy and that of the original equation. On the other hand, when the original nonlinear dispersive

system does not present an exact solitary–wave solution, the series will not truncate. In this case, a secular–free expansion can still be obtained and the process of returning to the laboratory coordinates can be made order–by–order at any higher order, implying in a sucessive solitary–wave velocity renormalization.[4,8]

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REFERENCES

[a] On leave from Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405–900 São Paulo SP, Brazil.

[1] V. E. Zakharov and A. B. Shabat, Func. Anal. Appl. **8**, 226 (1974).

[2] F. Calogero, in *What Is Integrability?* ed. by V. E. Zakharov (Springer, Berlin, 1991).

[3] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon and H. C. Morris, *Solitons and Nonlinear Waves* (Academic, London, 1982).

[4] R. A. Kraenkel, M. A. Manna and J. G. Pereira, J. Math. Phys. **36**, 307 (1995).

[5] T. Taniuti, Suppl. Prog. Theor. Phys. **55**, 1 (1974).

[6] P. D. Lax, Comm. Pure Appl. Math. **21**, 467 (1968).

[7] G. Sandri, Nuovo Cimento **36**, 67 (1965).

[8] Y. Kodama and T. Taniuti, J. Phys. Soc. Jpn, **45**, 298 (1978).